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Second-order two-dimensional solution for the drainage of recharge based on Picard's iteration technique: A generalized Dupuit-Forchheimer equation

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[1] Aquifer recharge is one of the most important problems in hydrology from both theoretical and practical points of view. One of the most widely accepted methods to deal with this problem is the use of the Dupuit-Forchheimer theory. This theory assumes that the water table is almost horizontal, the vertical velocity is zero, and the horizontal velocity is uniform with depth. Surfaces of seepage are not considered. Despite these strong limitations the theory is applied, and success is frequently found in many cases despite its fundamental assumptions being violated. In this work an approximate 2-D solution to the problem is sought on the basis of Picard's iteration technique, from which a second-order differential equation for recharge problems is found. On the basis of this solution, a modified, analytical Dupuit-Forchheimer (DF) ellipse is found which compares favorably with the full 2-D solution of the problem. The analytical developments of this theory provide a generalized DF theory which permits as an outcome the analytical determination of the surface of seepage.

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1. Introduction

[2] Aquifer recharge is one of the most relevant problems of Hydrogeology today, especially in phreatic aquifers [*Jaeger*, 1956; *Bear*, 1972]. The solution of the water flow problem requires the use of the Laplacian for the potential and stream functions subjected to the different boundary conditions. The flow equation and the free surface boundary condition are nonlinear, which implies in most cases a numerical solution [*Bear*, 1972; *Serrano*, 1995; *Rushton and Youngs*, 2010]. An alternative to this numerical solution is the adoption of Dupuit-Forchheimer hypotheses [*Dupuit*, 1863] (henceforth referred to as DF theory), which allow a simpler solution. The classical DF does not consider the existence of seepage surfaces that reduces the accuracy of the estimation of the water table position under aquifer recharge conditions [*Youngs*, 1990; *Knight*, 2005]. This fact prompted many researchers to search for alternative, still 1-D, improved models [*Serrano*, 1995; *Knight*, 2005; *Castro-Orgaz*, 2011a, 2011b]. *Knight* [2005] and *Castro-Orgaz* [2011a, 2011b] presented more correct 1-D models expressed as differential equations for the water table position requiring numerical handling.

[3] For the particular case of soil water drainage under steady recharge, *Rushton and Youngs* [2010] demonstrated that the standard first-order ordinary differential equation arising from the DF theory compares well with water table elevations deduced from a complete 2-D numerical integration of the Laplace equation if the chosen boundary condition is the seepage face. However, this result must be theoretically discussed and highlighted. *Castro-Orgaz* [2011a] found that the streamline curvature and inclination effects did not affect the water table position for flow to drains. This analysis confirmed that the classical DF equation is a good approach for some water table studies, even though the model did not incorporate the aquifer recharge intensity, thereby precluding a general conclusion.

[4] The classical DF theory assumes that the water table height at the outflow section of a drainage area is identical to the water level there, equivalent to neglecting the surface of seepage [*Bear*, 1972]. The DF theory assumes a hydrostatic pressure variation with depth, which is stated to be limited to zones of the water table where its slope is small. Further, for drainage problems, this is stated to be incomplete because the water table is not a streamline, and it is also necessary to ensure that the vertical velocity is nearly zero [*Bear*, 1972]. The idealized flow condition is therefore an almost horizontal water table where velocity is nearly horizontal and uniform with depth. But what happens if the water table curvature is large although the slope is nearly zero? And if the vertical velocity is nearly zero but the horizontal velocity profile is markedly variable with depth?

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These situations indicate that several aspects of the DF are obscure and, at best, only partially addressed in the literature.

[5] The purpose of this work is to critically assess the validity of the DF theory in the case of aquifer recharge. A novel solution based on Picard iteration method will be adapted to incorporate the effect of the local discharge intensity, and horizontal and vertical velocity distributions on the water table shape. As an application, two important problems will be addressed: soil drainage with symmetrically downstream boundaries and the toe drain lying on an impermeable stratum.

2. Picard's Iteration Technique: Second-Order 2-D Solution for the Water Table Analysis in an Aquifer Under a Uniform Steady Recharge

[6] The Cauchy-Riemann conditions for steady groundwater 2-D flow in a homogeneous isotropic aquifer are [*Jaeger*, 1956; *Bear*, 1972]

$$u = -K\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial z} \tag{1}$$

$$v = -K\frac{\partial \phi}{\partial z} = +\frac{\partial \psi}{\partial x}. \tag{2}$$

[7] The velocity components in the Cartesian x and z directions are u and v, respectively, the constant hydraulic conductivity is K, $\phi = p/\gamma + z$ is the water potential or hydraulic head, with pressure p and specific gravity of water γ, and ψ is the stream function.

[8] The equations (1) and (2) can be used iteratively to compute ϕ and ψ, and, consequently, u and v starting with an initial function for each variable [*Matthew*, 1991]. The method is well known in open channel hydraulics, and in the solution of other nonlinear groundwater flow equations [*Celia et al.*, 1990; *Mehl*, 2006]. The iterative cycle will be described in detail to find a second-order differential equation for the water table with recharge.

[9] The iteration starts with a first-order approximation for the horizontal component of the velocity, u, as the uniform velocity profile

$$u^{(1)} = \frac{q}{h}. \tag{3}$$

Here $q = q(x)$ is unit discharge which is limited hereafter to a linear function of x, i.e., $q' = \text{const}$ and $q'' = 0$, with primes denoting derivatives with respect to x. The stream function ψ to first-order accuracy follows from equation (1) as

$$\psi^{(1)} = -\int u^{(1)} dz = -\frac{qz}{h}, \tag{4}$$

which satisfies the boundary conditions of ψ as $\psi = 0$ at $z = 0$ and $\psi = -q$ at $z = h$. The first-order approximation for the vertical velocity component, v, from equation (2), is then

$$v^{(1)} = \frac{\partial \psi^{(1)}}{\partial x} = -\left(q' - \frac{qh'}{h}\right)\frac{z}{h}. \tag{5}$$

With $v^{(1)}$ introduced into equation (2), the first-order potential function $\phi^{(1)}$ is found by integration as

$$-K\phi^{(1)} = \int v^{(1)} \mathrm{d}z = -\left(q' - \frac{qh'}{h}\right)\frac{z^2}{2h} + f(x). \tag{6}$$

The function $f(x)$ is unknown at this stage. The second-order approximation, $u^{(2)}$, for the horizontal component of the velocity u follows from equation (1) and the x derivative of equation (6) as

$$u^{(2)} = -K\frac{\partial \phi^{(1)}}{\partial x} = \left[\frac{q}{h^2}\left(-\frac{h'^2}{h} + \frac{h''}{2}\right) + \frac{q'h'}{h^2}\right]z^2 + f'. \tag{7}$$

Integration of equation (7) with respect to z leads to a second-order expression for the stream function ψ as

$$\psi^{(2)} = -\int u^{(2)} \mathrm{d}z = -\left[\frac{q}{h^2}\left(-\frac{h'^2}{h} + \frac{h''}{2}\right) + \frac{q'h'}{h^2}\right]\frac{z^3}{3} - f' z. \tag{8}$$

An expression for f' can now be found by using the boundary condition at the water table, $\psi = -q$ at $z = h$, leading to

$$-q = -\left[\frac{q}{h^2}\left(-\frac{h'^2}{h} + \frac{h''}{2}\right) + \frac{q'h'}{h^2}\right]\frac{h^3}{3} - f' h. \tag{9}$$

The derivative of function f is

$$f' = \frac{q}{h} - \frac{1}{3}\left[q\left(-\frac{h'^2}{h} + \frac{h''}{2}\right) + q'h'\right]. \tag{10}$$

Inserting the expression for f' into equation (7), yields a second approximation for u as

$$u^{(2)} = \frac{q}{h} + \left[\left(\frac{z}{h}\right)^2 - \frac{1}{3}\right]\left[q\left(\frac{h''}{2} - \frac{h'^2}{h}\right) + q'h'\right]. \tag{11}$$

The horizontal component of velocity at the water table to second-order accuracy, designated u_s, is

$$u_s = \frac{q}{h}\left(1 + \frac{hh''}{3} - \frac{2h'^2}{3} + \frac{2q'h'h}{3q}\right). \tag{12}$$

The value of function $f(x)$ can be deduced from equation (6) imposing the conditions $z = h$, where $p = 0$, and $\phi = h$:

$$f(x) = -Kh + \frac{1}{2}(q'h - qh') \text{ With } f' = -Kh' - qh''. \tag{13}$$

Substituting this expression for f back into equation (6) produces

$$-K\phi^{(1)} = -Kh + \frac{1}{2}(qh' - q'h)\left[\left(\frac{z}{h}\right)^2 - 1\right]. \tag{14}$$

Equation (14) then gives an alternative expression for the second-order approximation to the horizontal component of velocity u as

$$u^{(2)} = -K\frac{\partial \phi^{(1)}}{\partial x} = -Kh' - \frac{qh''}{2} + \left(\frac{qh''}{2} + q'h' - \frac{qh'^2}{h}\right)\left(\frac{z}{h}\right)^2. \tag{15}$$

Equation (15) at the water table yields

$$u_s = -Kh' + q'h' - \frac{qh'^2}{h}, \tag{16}$$

also obtained from equation (7) and the second of equations (13). Equating equations (12) and (16) produces an ordinary differential equation for h:

$$\frac{q}{h}\left(1 + \frac{hh'' + h'^2}{3} - \frac{q'h'h}{3q}\right) + Kh' = 0. \tag{17}$$

[10] Equation (17) is the second-order solution for the drainage problem obtained by the Picard iteration technique. For no recharge, equation (17) simplifies to the Dupuit-Fawer equation obtained by *Castro-Orgaz* [2011a, 2011b], expressing the Laplacian in curvilinear coordinates. Equation (17) is therefore a generalized result for the water table equation in Cartesian coordinates.

3. The Drainage of Recharge Problem

[11] For the case considered in Figure 1 the horizontal seepage discharge is $q(x) = Nx$, and equation (17) may be rearranged in the form

$$\left(1 + \frac{hh'' + h'^2}{3} - \frac{h'h}{3x}\right) + \frac{K}{N}\frac{h'h}{x} = 0. \tag{18}$$

After some mathematical manipulation it becomes

$$\left[1 + \frac{x}{3}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{h'h}{x}\right)\right] + \frac{K}{N}\frac{h'h}{x} = 0. \tag{19}$$

Introducing the variable $Z = hh'/x$, enables equation (19) to be recognized as a first-order ordinary differential equation with separated variables

$$\left(1 + \frac{x}{3}\frac{\mathrm{d}Z}{\mathrm{d}x}\right) + \frac{K}{N}Z = 0. \tag{20}$$

The solution in terms of an arbitrary constant c_1 is

$$1 + \frac{K}{N}Z = c_1 x^{-3K/N}. \tag{21}$$

Substituting for Z enables equation (21) to be rewritten as

$$hh' = \frac{N}{K}\left(c_1 x^{1-3K/N} - x\right), \tag{22}$$

which may be integrated with respect to x to give

$$\frac{1}{2}h^2 = \frac{N}{K}\left[\frac{c_1}{\left(2 - 3\frac{K}{N}\right)}x^{2-3K/N} - \frac{1}{2}x^2\right] + c_2. \tag{23}$$

To determine the constants c_1 and c_2, use is made first of the required symmetry condition $h' = 0$ at $x = 0$ in equation (22). Here it can be seen that c_1 depends on the value of K/N.

[12] When the exponent of x is positive, $h' = 0$ automatically, regardless of the value of c_1. When the exponent is not positive, i.e., $2 - 3K/N \leq 0$ or $N/K \leq 3/2$, then $c_1 = 0$. Hereafter, this condition $c_1 = 0$ will be used,

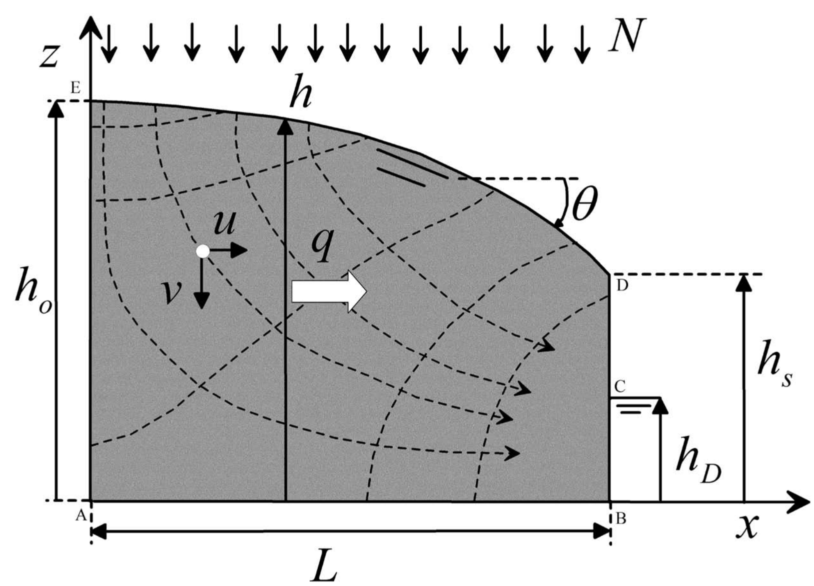

Figure 1. Streamlines and equipotential lines in a phreatic aquifer above an impervious horizontal layer under a uniform and steady recharge of intensity N. The local height of the water table is h.

because from numerical calculations to be presented, N/K values greater than approximately 0.6, and the same critical quantity $2 - 3K/N < 0$ arising in an estimation of h_o, show that the second-order Picard approximations diverge considerably from examples of 2-D calculations, and are no longer appropriate. The constant c_2 is determined from equation (23) by imposing the requirement that at $x = 0$, $h = h_o$ at the water table, h_o yet to be determined. Immediately, $c_2 = h_o^2/2$ and equation (23) becomes

$$h^2 = h_o^2 - \frac{N}{K}x^2. \tag{24}$$

[13] The solution of equation (24) coincides with the result of the Dupuit-Forchheimer analysis. This is a fortuitous coincidence because the velocity field associated with equation (30) is neither uniform along the horizontal (from equation (11)),

$$u = \frac{q}{h}\left[1 + \left(\frac{hh''}{2} - h'^2 + \frac{q'h'h}{q}\right)\left(\frac{3\eta^2 - 1}{3}\right)\right], \eta = \frac{z}{h}, \tag{25}$$

nor along the vertical (from equation (5)),

$$v = \frac{q}{h}\left(h' - \frac{q'h}{q}\right)\eta. \tag{26}$$

Equation (24) is a 2-D solution for the problem of recharge, which is equation (21) or equation (22) with $c_1 = 0$, rewritten as,

$$\frac{K}{N}\frac{hh'}{x} = -1, \tag{27}$$

which, when compared with equation (18), is subjected to the peculiar relationship

$$hh'' + h'^2 = \frac{h'h}{x}. \tag{28}$$

Equation (27) implies that the flow rate q is exact for this problem as

$$q = Nx = -Khh'. \tag{29}$$

The velocity components from equations (25) and (26) are, using the condition given by equation (28),

$$u = \frac{Nx}{h}\left[1 + \left(\frac{hh'}{x} - h'^2\right)\left(\frac{3\eta^2 - 1}{2}\right)\right] \tag{30}$$

$$v = \frac{Nx}{h}\left(h' - \frac{h}{x}\right)\eta. \tag{31}$$

[14] Equation (29) may be accepted as the exact 2-D governing equation to second-order accuracy for the drainage of recharge problem. This equation yields the generalized water table profile given by equation (24). The equation does not suffer from the limitations of the classical Dupuit-Forchheimer model, implying an almost horizontal water table. It is valid for curvilinear flow with recharge. In addition, the horizontal velocity u is not uniform, and the vertical velocity is not zero, as seen in equations (30) and (31). Equation (24) applies in regions of large vertical velocity. For the limiting case at $x = 0$, equations (30) and (31) produce

$$u = 0, \qquad v = -N. \tag{32}$$

Equation (32) implies a vertical flow. This is a flow condition deduced from the flow equations and therefore implicit from equation (24).

3.1. Boundary Condition for the 2-D Model

[15] Equation (24) requires a boundary value for h_o. This value needs to be accurate to obtain precise water table elevations. Nothing is assumed about the existence of seepage faces. The value of h_o may be taken from the complete 2-D solution of the groundwater flow problem. The classical assumptions of the Dupuit-Forchheimer theory about the surfaces of seepage are unnecessary, and very limited. The model given by equation (24) is exact to second order. This result agrees with the findings of *Rushton and Youngs* [2010] of good agreement between the Dupuit-Forchheimer equation and their 2-D solution of Laplace's equation, if the boundary condition for the former was taken as the seepage face height. The usefulness of a 1-D approach as compared with the complete 2-D solution was, however, not theoretically justified. To complete the model presented here an estimate of the boundary condition h_o is required.

3.2. Estimate of h_o Using Green's Second Identity

[16] Green's second identity [*Courant and Hilbert*, 1962] for two arbitrary functions, w and ϕ, in 2-D reduces, when they are both regular harmonic functions, to [*Huard de la Marre*, 1956; *Chapman*, 1957a, 1957b, 2003]

$$\oint w\frac{\partial \phi}{\partial n}ds = \oint \phi\frac{\partial w}{\partial n}ds. \tag{33}$$

The derivatives are evaluated in the normal direction n external to the closed surface, and s is the curvilinear distance along its perimeter. For the case depicted in Figure 1, which defines θ as the angle between free surface and the x axis, Table 1 summarizes each term along the boundary [*Chapman*, 2003] using a value of $w = x$.

[17] The kinematic boundary condition at the free surface is [*Bear*, 1972; *Youngs* and *Rushton*, 2009; *Rushton and Youngs*, 2010]

$$\frac{q}{K}\cos\theta = \frac{\partial \phi}{\partial x}\sin\theta + \frac{\partial \phi}{\partial z}\cos\theta = \frac{\partial \phi}{\partial n} \tag{34}$$

Table 1. Evaluation of Arguments for Green's Second Identity (Equation (33))

Line	w	$K\frac{\partial\phi}{\partial n}$	$K\phi$	$\frac{\partial w}{\partial n}$
AB	x	0	unknown	0
BC	L	$-u$	Kh_D	1
CD	L	$-u$	Kz	1
DE	x	equation (39)	Kh	$-\sin\theta$
EA	0	0	unknown; use equation (40)	-1

This equation represents a mass conservation across the water table surface. This equation is required in equation (33) for the curvilinear integral along the water table surface. The form of the potential equation, ϕ, is unknown along the boundaries AB and EA. The contribution of the former is zero, whereas the latter needs to be evaluated.

[18] Equation (14) at $x = 0$ where $h' = 0$ yields

$$\phi(0, z) = h_o - \frac{q'}{2Kh_o}(h_o^2 - z^2). \tag{35}$$

Equation (35) is the identical distribution proposed by *Chapman* [2003], who assumed the streamlines near $x = 0$ to be of the type $xz =$ const. and the equipotentials $x^2 - z^2$ const. Therefore, the analysis based on Picard iteration confirms that equation (35) is generally valid to this order of accuracy. In addition, the role of the kinematic boundary condition for ϕ is clearly specified with equation (34). Inserting each term of Table 1 into equation (33) finally yields [*Chapman*, 2003]

$$h_o^2 = \left(h_D^2 + \frac{N}{K}L^2\right)\left(1 - \frac{2N}{3N}\right)^{-1}, N/K < 3/2. \tag{36}$$

It is noted that the denominator of equation (36) must not be zero i.e., $3K/N < 3/2$ and is also a critical relation required to ensure that $c_1 = 0$ for equation (23). Equation (36) is therefore the generalized second-order result for the upstream boundary condition h_o. It was originally obtained by *Chapman* [2003] assuming a parabolic distribution given by equation (35). The current results demonstrate that this function arises from a second-order approximation to the 2-D problem. Equation (36) was also obtained by *Knight* [2005] with another approximate theory. However, the developments presented herein demonstrate that equation (36) is a more general approximation to the 2-D problem. Using equation (36), the generalized result for the water table profile is, from equation (24),

$$h^2 = \left[h_D^2 + \frac{N}{K}(L^2 - x^2) + \frac{2}{3}\left(\frac{N}{K}\right)^2 x^2\right]\left(1 - \frac{2N}{3K}\right)^{-1}, N/K < 3/2. \tag{37}$$

[19] As far as the authors are aware, equation (37) has not been presented before in the groundwater literature. An immediate consequence of equation (37) is the evaluation of h_s, the height of the surface of seepage at $x = L$:

$$h_s^2 = \left[h_D^2 + \frac{2}{3}\left(\frac{N}{K}\right)^2 L^2\right]\left(1 - \frac{2N}{3K}\right)^{-1}, N/K < 3/2. \tag{38}$$

4. Comparison of Picard Iteration With 2-D Solutions of Groundwater Flow Problems

[20] With second-order truncation of Picard iteration and lack of proof of convergence for a large number of iterations it is important to make comparisons with 2-D problems solved with 2-D solution procedures. Two problems are selected. The first is the specification of different combinations of h_D/L, h_o/L, and N/K by *Hornung and Krueger* [1985] and *Rushton and Youngs* [2010]. The second problem is for a toe drain upon an infinite impermeable layer which *Engelund* [1951] solved analytically.

4.1. Comparison With the Results of *Rushton and Youngs* [2010]

[21] Equation (37) is compared in Figure 2 with the 2-D seepage water table results by *Rushton and Youngs* [2010] solving the complete equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \tag{39}$$

subjected to the following boundary conditions:

$$\begin{aligned}
&\frac{\partial\phi}{\partial z} = 0 && \text{on AB} \\
&\phi = h_D && \text{on BC} \\
&\phi = z && \text{on CD} \\
&\phi = h && \text{on DE} \\
&\frac{q}{K}\cos\theta = \frac{\partial\phi}{\partial x}\sin\theta + \frac{\partial\phi}{\partial z}\cos\theta && \text{on DE} \\
&\frac{\partial\phi}{\partial x} = 0 && \text{on EA}\ .
\end{aligned} \tag{40}$$

[22] The data was digitized from the published solution by *Rushton and Youngs* [2010]. Figure 2 indicates the good agreement between the numerical solution to the 2-D problem, equation (39) subjected to conditions (40), and the 2-D Picard solution to second-order accuracy given by equation (37).

[23] The present analytical result is a considerable advance over the numerical solution by *Castro-Orgaz* [2011a], who

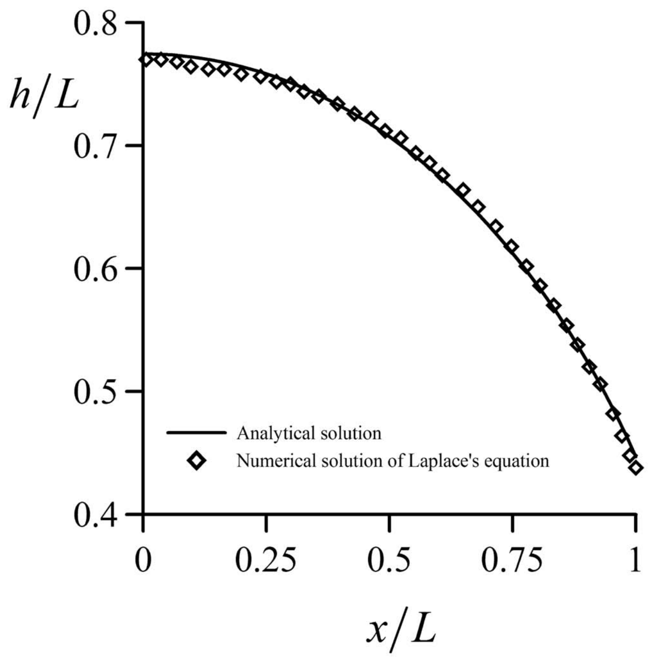

Figure 2. Recharge to symmetrically located downstream boundaries for $N/K = 0.4$ and $h_D/L = 0.2$: comparison of two-dimensional seepage water table results by *Rushton and Youngs* [2010] with equation (37).

estimated the water table elevation by an approximate solution of the Laplacian flow equation in curvilinear coordinates. Equation (37) provides, therefore, a practical and simple analytical formula for hydrologists working in phreatic aquifers.

[24] *Rushton and Youngs* [2010] presented numerical values for h_o/L and h_s/L from Laplacian simulations for two test cases: (1) $h_D/L = 0$ (Figure 3a) and (2) $h_o/L = 0.5\ N/K$ (Figure 3b). These are compared with estimates given by equations (36) and (38), respectively. The good agreement of the approximate 2-D model based on Picard iteration can be seen. Therefore, the empirical fittings to h_s/L proposed by *Rushton and Youngs* [2010] to find the free surface profile may be replaced by the analytical approach given by equations (36), (37), and (38). The Picard iteration method demonstrates that equation (24), the so-called Dupuit-Forchheimer ellipse for the drainage, is NOT limited to a 1-D model. It is an exact 2-D solution to second-order accuracy, justified by the success of its comparison with the 2-D results from the numerical solution of the flow equation.

[25] *Hornung and Krueger* [1985] presented numerical simulations for the water table for $N/K = 0.4$ and 0.1 and $h_D/L = 0$. Their results are compared in Figure 4a with equation (37) and show good agreement. The values for h_o/L and h_s/L from *Hornung and Krueger* [1985] are compared against estimates given by equations (36) and (38), respectively, in Figure 4b, resulting in good agreement for $N/K < 0.6$. Youngs' drainage inequality [*Youngs*, 1965]

$$\frac{N}{K} \leq h_o^2 \leq \frac{N}{K}\left(1-\frac{N}{K}\right)^{-1}, \tag{41}$$

is plotted as a shadow domain in Figure 4c, where the results from *Hornung and Krueger* [1985] and equation (36) demonstrate that the latter satisfies equation (41).

4.2. Comparison With Engelund's Hodograph Plane Theory

[26] *Engelund* [1951] considered the problem of drainage of recharge for a toe drain overlying an impermeable stratum (Figure 5). He applied the hodograph transformation to solve the 2-D problem, finding that the water table elevation is

$$h^2 = \frac{N}{K}(L^2 - x^2), \tag{42}$$

where L is the lateral distance to the point at which $h = 0$.

[27] *Youngs* [2012] has recently indicated that equations (24) and (42) may match fortuitously under different assumptions. Equation (24) was demonstrated to be a 2-D solution, and that its boundary condition should be a particular point solution of the 2-D problem. Setting $h = 0$ at $x = L$ in equation (24) gives

$$h_o^2 = \frac{N}{K}L^2. \tag{43}$$

After inserting h_o^2 into equation (24), then equation (42) is regained. In consequence, the 2-D solution obtained by Picard iteration to second-order accuracy is identical to the 2-D solution obtained by *Engelund* [1951], provided that the boundary condition used in equation (24) is also a point of the 2-D solution. Engelund's solution may be further

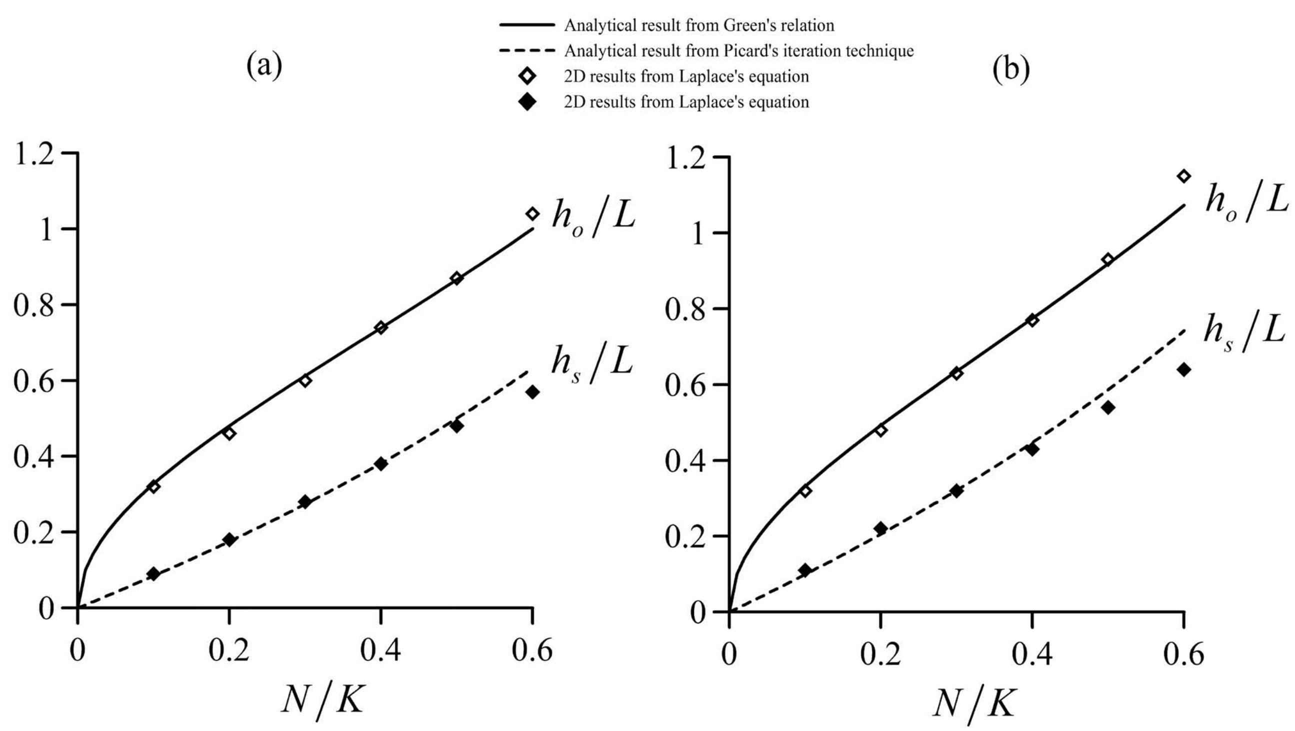

Figure 3. Comparison of the ratios h_o/L and h_s/L from the 2D solution of the Laplace equation [*Rushton and Youngs*, 2010] for two test cases; (a) $h_D/L = 0$, (b) $h_o/L = 0.5N/K$ against equations (36) and (38).

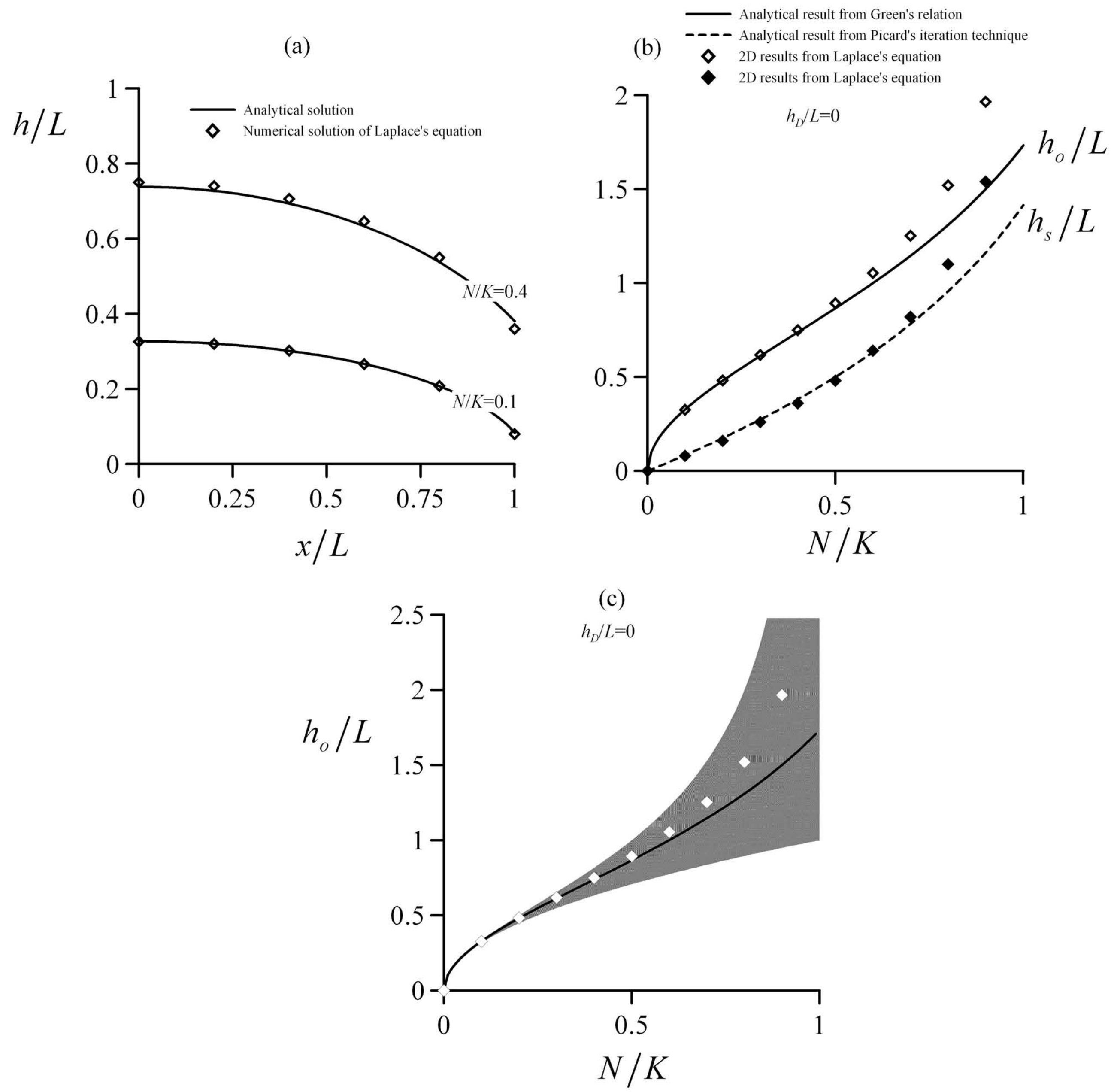

Figure 4. Recharge to symmetrically located downstream boundaries: (a) comparison of two-dimensional seepage water table results by *Hornung and Krueger* [1985] for $N/K = 0.4$ and 0.1 and $h_D/L = 0$ with equation (37), (b) comparison of h_o/L and h_s/L from the two-dimensional solution of the Laplace equation [*Hornung and Krueger*, 1985] for $h_D/L = 0$ versus equations (36) and (38), (c) equation (36) and 2-D data from *Hornung and Krueger* [1985] plotted in the drainage inequality diagram of *Youngs* [1965] (shaded area).

compared with the present theory in terms of stream and potential functions, from which the full 2-D problem is defined. Engelund's full 2-D solution is

$$x^2 - z^2 - \left(1 - \frac{N}{K}\right)^2 L^2 = \frac{\psi^2}{NK} - \frac{NK}{\psi^2} x^2 z^2 \quad (44)$$

$$x^2 - z^2 - \left(1 - \frac{N}{K}\right)^2 L^2 = -\frac{K\phi^2}{N} + \frac{N}{K\phi^2} x^2 z^2. \quad (45)$$

From our approach, equation (30) provides

$$\psi = -\int u \mathrm{d}z = -Nx\left[\eta + \left(\frac{hh'}{x} - h'^2\right)\left(\frac{\eta^3 - \eta}{2}\right)\right], \quad (46)$$

and equation (14) yields

$$\phi = h\left[1 + \frac{1}{2}\frac{N}{K}\left(1 - \frac{xh'}{h}\right)(\eta^2 - 1)\right]. \quad (47)$$

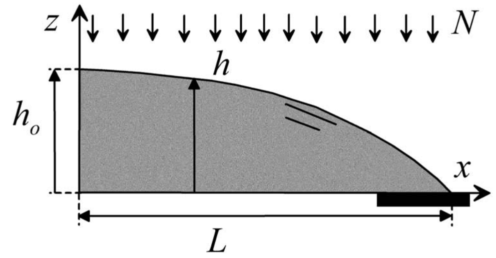

Figure 5. The saturated flow domain of an aquifer under a uniform recharge rate N with a toe drain resting on an impermeable layer [*Engelund*, 1951].

Using the water table differential equation $Khh' = -Nx$, equations (46) and (47) are reduced to

$$\tilde{\psi} = \eta\left[1 - \frac{\alpha}{2}(1 + \alpha r^2)(\eta^2 - 1)\right] \tag{48}$$

$$\tilde{\phi} = 1 + \frac{\alpha}{2}(1 + \alpha r^2)(\eta^2 - 1), \tag{49}$$

with the new parameter $\alpha = N/K$, new variable $r = x/h$, and new variables defined by $\tilde{\phi} = \phi/h$ and $\tilde{\psi} = \psi/(-Nx)$.

[28] Dividing equation (44) by h^2 produces a quadratic equation in $\tilde{\psi}^2$ whose coefficients involve parameters α and L/K and variables r and η. Dividing equation (45) by h^2 also produces a quadratic equation in $\tilde{\phi}^2$ with the same parameters and variables. Now if α and x/L are given, equation (42) then determines L/h so that $r = x/L \cdot L/h$. In this case there remain $\tilde{\psi}$-η and $\tilde{\phi}$-η relationships. Equations (44) and (45) are compared in Figure 5 with equations (48) and (49) for a drainage case with $N/K = 0.2$. A comparison for $\tilde{\psi}$ between both methods is presented at positions $x/L = 0.2$, 0.4, and 0.6 in Figures 6a, 6b, and 6c, respectively, and the same comparison for $\tilde{\phi}$ is made in Figures 6d, 6e, and 6f. It can be seen that the agreement between the full 2-D solution and Picard second-order approximation is very good for $\tilde{\psi}$ and acceptable for $\tilde{\phi}$. Deviations increase as the boundary condition $h = 0$ is approached, as expected from the mathematical approximation in equations (48) and (49). However, their accuracy is adequate for practical purposes.

5. Discussion

5.1. The Classical DF Theory

[29] The problem of drainage of recharge with symmetrically downstream boundaries is one of the basic and important problems in groundwater hydrology [*Kirkham*, 1967; *Rushton and Youngs*, 2010]. This problem may be tackled either using the complete 2-D solution of Laplace equation for the hydraulic head, or using the Dupuit-Forchheimer approximation. This approximate 1-D theory is presented in papers and books starting with a water table of small inclination, thereby leading to a flow with almost parallel, horizontal streamlines. The literature then traditionally calls for caution when using this theory, which is limited to [*Bear*, 1972] (1) an almost horizontal water table, (2) flows almost horizontal, meaning that the vertical velocity needs to be zero or very small, which invalidates the DF theory in the

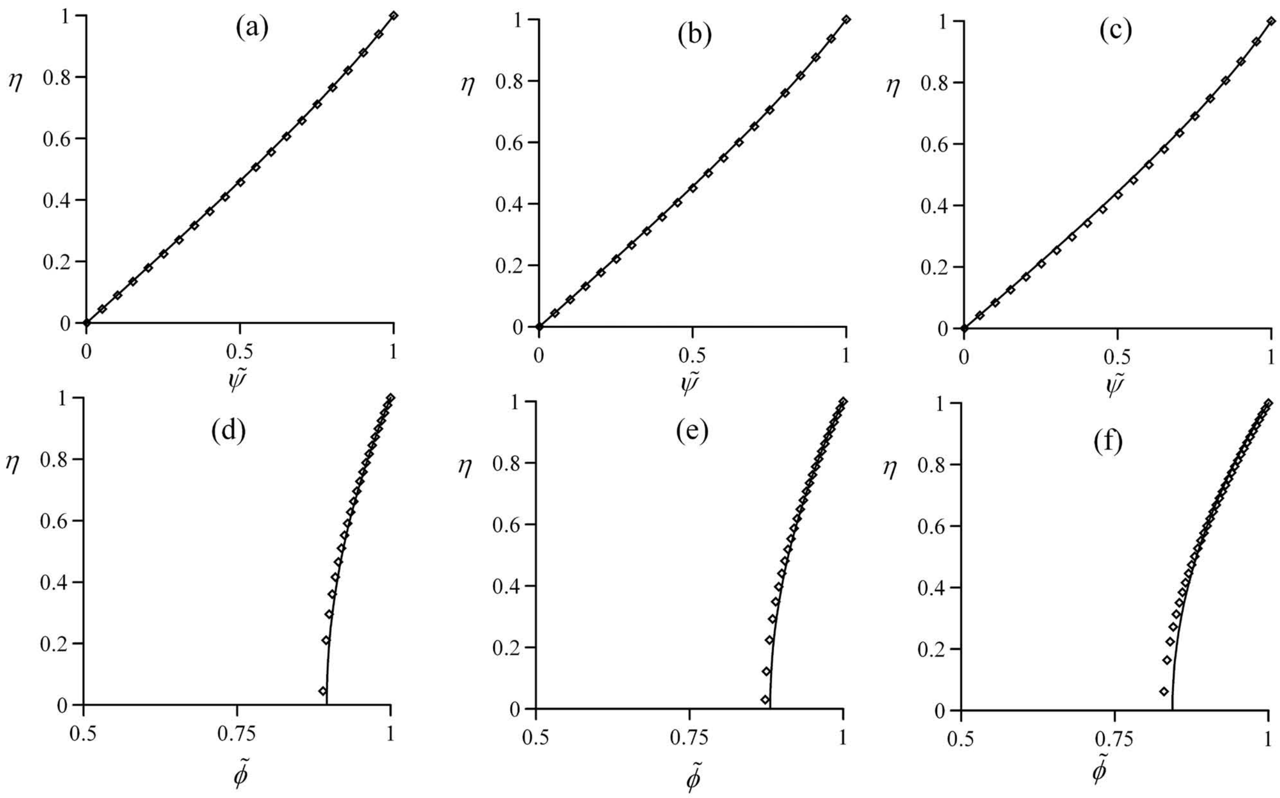

Figure 6. Comparison of results from equations (44) and (45) (diamonds) with those from equations (48) and (49) (lines) for $N/K = 0.2$ at positions (a, d) $x/L = 0.2$, (b, e) $x/L = 0.4$, and (c, f) $x/L = 0.6$.

case shown in Figure 1 at $x = 0$ because there although $h' = 0$, the vertical velocity is not zero and given by $v = -N$, and (3) the classical DF solution for the case of Figure 1 assuming that the water table reaches the water level h_D, which is equivalent to neglecting the surface of seepage.

5.2. A Revised DF Theory Based on the Full 2-D Solution

[30] The traditional DF approach (summarized above) was recently revised by *Rushton and Youngs* [2010], and the following points were outlined in their study.

[31] 1. The DF equation is a 1-D model which may be solved subjected to a boundary condition at some undefined position. *Rushton and Youngs* [2010] indicated that there is no reason why this should not be enforced as the downstream boundary condition.

[32] 2. Better water table estimations can be obtained if the surface of seepage is taken as this boundary condition.

[33] 3. For this task, a 2-D numerical model was used to solve the problem. Numerical results for the surface of seepage height were fitted to empirical relationships for their use in the classical DF differential equation.

[34] 4. No explanations are given of reasons why this approach is successful for comparisons with the 2-D model. Their proposed model also depends on the empirical relationships developed for the boundary condition at the surface of seepage.

5.3. Approximate 2-D Solution Based on Picard's Iteration Technique

[35] A simple method to obtaining higher-order solutions is the Picard iteration method [*Matthew*, 1991]. On the basis of the second-order solution with this technique, the following results were found.

[36] 1. The DF theory is not limited to flows with $h' \approx 0$. The local discharge relation $q = -Khh'$ is generally valid for curvilinear flows with recharge provided that there is a local equilibrium given by $hh'' + h'^2 = hh'/x$ with $N/K < 3/2$. This relation was found to be exact for the drainage of recharge with symmetrically located downstream boundaries (Figure 1). It means that the discharge relation $Nx = -Khh'$ is the exact differential equation governing the water table shape to second-order accuracy. It is associated with nonuniform horizontal and vertical velocities (equations (30) and (31)).

[37] 2. The differential equation yields a generalized water table profile function with a suitable boundary condition to be determined. On the basis of a 2-D computation using the second Green identity for harmonic functions, the boundary condition at $x = 0$ was found. The result is a drainage ellipse, similar to the classical DF ellipse, but with the correct boundary condition based on the 2-D results. It is therefore not necessary to use an empirical relationship for the seepage height h_s as a boundary condition. Instead, an analytical 2-D result for h_D is proposed.

[38] 3. The new DF ellipse (equation (37)) provides an analytical expression for the height of the surface of seepage (equation (38)). It demonstrates that the DF equation has nothing implicit about the existence of the surfaces of seepages if the boundary condition is correctly accounted for. The existence of a surface of seepage is demonstrated to be an outcome of the model.

[39] 4. The new result for an improved DF theory given by Picard's iteration technique is different from the DF theory as presented by *Kirkham* [1967]. He obtained as a result of his DF "soil" a flow net composed of curvilinear streamlines and vertical equipotentials, thereby violating the normal intersection of both families of curves. In contrast, our second-order results has nothing implicit about these conditions, and orthogonality of the flow net is preserved.

6. Conclusions

[40] The following conclusions are derived from this work.

[41] 1. Using the Picard iteration technique, it was found that the classical DF theory is not limited to almost horizontal flows with negligible vertical velocity for the problem of recharge.

[42] 2. The DF differential equation is demonstrated to be the exact equation governing the water table to second-order accuracy.

[43] 3. On the basis of Green's second identity, a suitable 2-D boundary condition at $x = 0$ was found, thereby producing a modified DF drainage ellipse which provides an analytical expression for the height of the surface of seepage.

[44] 4. The DF improved drainage ellipse is shown to be associated with nonuniform horizontal velocities and nonzero vertical velocities.

Notation

$c_{1,2}$	constants of Integration.
f	function, m.
h	water table height, m.
K	hydraulic conductivity, m s^{-1}.
n	coordinate normal to external boundary, m.
N	recharge, m s^{-1}.
q	unit discharge, m^2 s^{-1}.
L	half separation between drainage ditches, m.
s	curvilinear coordinate along external boundary, m.
u	horizontal velocity, m s^{-1}.
v	vertical velocity, m s^{-1}.
w	a harmonic function, m^2 s^{-1}.
x	horizontal distance, m.
z	elevation, m.
γ	specific weight of water, N m^{-3}.
θ	angle of water table with horizontal, rad.
ψ	stream function, m^2 s^{-1}.
ϕ	water potential, m.
	Subscripts
o	relative to upstream section.
D	relative to tail water section.
s	relative to seepage face; also to free surface.

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